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A TWO-VARIABLE INTERLACE POLYNOMIAL

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We introduce a new graph polynomial in two variables. This "interlace" polynomial can be computed in two very different ways. The first is an expansion analogous to the state space expansion of the Tutte polynomial; the significant differences are that our expansion is over vertex rather than edge subsets, and the rank and nullity employed are those of an adjacency matrix rather than an incidence matrix.

The second computation is by a three-term reduction formula involving a graph pivot; the pivot arose previously in the study of interlacement and Euler circuits in four-regular graphs.

We consider a few properties and specializations of the two-variable interlace polynomial. One specialization, the "vertex-nullity interlace polynomial", is the single-variable interlace graph polynomial we studied previously, closely related to the Tutte–Martin polynomial on isotropic systems previously considered by Bouchet. Another, the "vertex-rank interlace polynomial", is equally interesting. Yet another specialization of the two-variable polynomial is the independent-set polynomial.

1. The interlace polynomial

In [2,1], we introduced a single-variable "interlace" graph polynomial. It emerged that the interlace polynomial could be regarded as a special case of the Tutte–Martin polynomial of an isotropic system, as discussed briefly in Section 4 here and more fully in [1]. We defined the polynomial by a recurrence relation, and Balister, Bollobás, Cutler and Pebody [4] used a property of the nullities of the adjacency matrices of the graphs in question to resolve

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a conjecture posed in [2]. The linear-algebraic approach of [4], extended to embrace the matrix ranks as well as nullities, led us to the two-variable polynomial introduced here. The two-variable interlace polynomial appears to be something entirely new; in particular, we are aware of no two-variable versions of the Tutte–Martin polynomial or other closely related polynomials. The interlace polynomial is an entirely different object from the Tutte polynomial, but the two have an extremely interesting structural similarity which immediately suggests a family of additional polynomials for further exploration. We will now define the two-variable interlace polynomial.

Given a graph G with vertex set $V(G)$, for any subset $S \subseteq V(G)$, let $G[S]$ be the subgraph of G induced by S. We allow graphs with loops on their vertices, and we also allow the *null graph* with no vertices, but we do not allow multiple loops or multiple edges. Write $\mathcal{G}$ for the set of graphs including the null graph.

For a matrix A over $\mathbb{F}_2$, let $n(A)$ be the nullity of A and $r(A)$ its rank. Abusing notation slightly, for a graph G, $n(G)$ and $r(G)$ will denote the nullity and rank of its adjacency matrix, so $n(G) + r(G) = |V(G)|$. (For example, if S is the empty set then $G[S]$ is the null graph of rank and nullity 0.) We remark that for loopless graphs G, $r(G)$ is always even, as the rank of a zero-diagonal symmetric matrix.

We define the two-variable interlace polynomial $q(G; x, y)$ of a graph G of order n as a sum of 2^n terms:

$$q(G; x, y) = \sum_{S \subseteq V(G)} (x-1)^{r(G[S])}(y-1)^{n(G[S])}, \tag{1}$$

the sum taken over all subsets including $S = \emptyset$ and $S = V(G)$. For convenience we define the monomial

$$m(H) = (x-1)^{r(H)}(y-1)^{n(H)},$$

so that

$$q(G) = \sum_{S} m(G[S]). \tag{2}$$

This “state space” expansion of the two-variable interlace polynomial may be seen as an analogue of the Tutte polynomial given as

$$T(G; x, y) = \sum_{F \subseteq E} (x-1)^{r(E)-r(F)}(y-1)^{n(F)}. \tag{3}$$

The significant differences are that our sum is over induced subgraphs of G (as given by vertex subsets) rather than arbitrary subgraphs on the full vertex set (as given by edge subsets), and our rank and nullity are those of the subgraph's adjacency matrix rather than its incidence matrix. (In the context of the Tutte polynomial, a graph's rank is normally defined as the number of vertices minus the number of components, and it is easy to check that this is the $\mathbb{F}_2$-rank of the incidence matrix.) That our subgraph rank appears positively rather than subtracted from the rank of the whole is not significant, as it can be adjusted by a change of variables. That is, if one prefers the polynomial $\sum_{S\subseteq V(G)}(x-1)^{r(G)-r(G[S])}(y-1)^{n(G[S])}$, it is just $(x-1)^{r(G)}q(\frac{x}{x-1},y)$.

A surprising basic property of this polynomial is that for loopless graphs it satisfies a three-term *reduction formula*, as per Theorem 3, and for looped graphs, a pair of reductions, per Theorem 6; we prove these results in Section 2. In Section 3 we show a pair of properties of the two-variable interlace polynomial. In Section 4 we describe the polynomial's specializations to the vertex-rank polynomial and the vertex-nullity polynomial, and in Section 5, its specialization to the independent-set polynomial. We calculate the polynomial on some basic graphs in Section 6. We conclude in Sections 7 and 8 with thoughts on generalizations of the polynomial, some of which seem likely to prove interesting, and with some open problems.

We now proceed to the interlace polynomial's reduction formula.

2. Reduction formula

We begin by showing a reduction involving a "pivoting" operation on an edge ab of G for which both a and b are loopless. It is natural to start with this case because it provides a recursive definition of $q(G)$ for the natural class of loopless graphs G. In the following subsection, we show how to reduce on a looped vertex a of G, completing a recursive definition of $q(G)$ for arbitrary graphs.

2.1. The pivot and reduction

As in [2,1,9], and related to Kotzig's transformations on Euler tours [16], for a graph G and an ordered pair $ab=(a,b)$ of distinct vertices of G, we define the *pivot operation on ab* mapping G into G^{ab} as follows. We say that two vertices x,y of G are *distinguished* by $\{a,b\}$ if $x,y\notin\{a,b\}$ and x,y have distinct non-empty neighborhoods in $\{a,b\}$. Let G^{ab} be the graph with vertex set $V(G)$ in which xy is an edge if either $xy\notin E(G)$ and x and y are distinguished by $\{a,b\}$, or else $xy\in E(G)$ and x and y are not distinguished by $\{a,b\}$.

Let us spell out this definition in detail. Partition the vertices other than a and b into four classes:

(C1) vertices adjacent to both a and b;
(C2) vertices adjacent to a alone;
(C3) vertices adjacent to b alone; and
(C4) vertices adjacent to neither a nor b.

Definition 1 (Pivot). A graph G is pivoted on vertices a,b to obtain G^{ab} as follows. For any vertex pair xy where x is in one of the classes (C1)–(C3) and y is in a different class (C1)–(C3), the pair xy is "toggled": if it is an edge of G it is not an edge of G^{ab}, and if it is not an edge of G then it is an edge of G^{ab}. All other pairs of vertices are adjacent in G^{ab} iff they are adjacent in G.

Trivially, pivoting and restriction to a subgraph satisfy a commutative law: for any $S \subseteq V(G)$ with $a,b \in S$,

$$G^{ab}[S] = (G[S])^{ab}. \tag{4}$$

Although pivoting is defined for any vertex pair ab, we shall only exploit it in cases where ab is an edge and a and b are both loopless.

We shall write the adjacency matrices of G and G^{ab} with rows and columns put into six groups according to their relations to a and b. The first group consists of a alone, and the second of b alone; groups three to six are the four classes above. Write $\mathbf{1}$ for an all-1 row or column vector of whatever dimension and likewise $\mathbf{0}$ for an all-0 vector. Then, for a graph G with loopless vertices a and b and having an edge ab, the adjacency matrix of G is of the form

$$A(G) = \begin{pmatrix} 0 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & M_{11} & M_{12} & M_{13} & M_{14} \\ \mathbf{1} & \mathbf{0} & M_{21} & M_{22} & M_{23} & M_{24} \\ \mathbf{0} & \mathbf{1} & M_{31} & M_{32} & M_{33} & M_{34} \\ \mathbf{0} & \mathbf{0} & M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}, \tag{5}$$

where in all cases M_{ji} is the transpose of M_{ij}. Then for such a graph the adjacency matrix of G^{ab} is

$$A(G^{ab}) = \begin{pmatrix} 0 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & M_{11} & M_{12}^c & M_{13}^c & M_{14} \\ \mathbf{1} & \mathbf{0} & M_{21}^c & M_{22} & M_{23}^c & M_{24} \\ \mathbf{0} & \mathbf{1} & M_{31}^c & M_{32}^c & M_{33} & M_{34} \\ \mathbf{0} & \mathbf{0} & M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} \tag{6}$$

where M^c denotes the complement of the matrix M.

Lemma 2. *For any graph G with an edge ab, with a and b both loopless, $r(G-a)=r(G^{ab}-a)$ and $r(G)=r(G^{ab}-a-b)+2$; equivalently, $n(G-a)=n(G^{ab}-a)$ and $n(G)=n(G^{ab}-a-b)$.*

Proof. With the adjacency matrix $A=A(G)$ represented as in (5), add row 2 to each row in the 3rd and 4th groups. Repeat the same operations on columns to give a matrix

$$A' = \begin{pmatrix} 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & M_{11} & M_{12}^c & M_{13}^c & M_{14} \\ \mathbf{0} & \mathbf{0} & M_{21}^c & M_{22} & M_{23}^c & M_{24} \\ \mathbf{0} & \mathbf{1} & M_{31}^c & M_{32}^c & M_{33} & M_{34} \\ \mathbf{0} & \mathbf{0} & M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}.$$

Note that, outside of the neighborhoods of a and b (the first two rows and columns), A' is the adjacency matrix of G^{ab}; moreover, since these linear operations are invertible, $r(A)=r(A')$.

To prove the first assertion, discard the first row and column of A to yield $A\backslash a$ and similarly that of A' to obtain $A'\backslash a$. Since they did not use row or column 1 (vertex a), the same linear transformations as before map $A\backslash a$ to $A'\backslash a$, showing that $r(A\backslash a) = r(A'\backslash a)$. But $A\backslash a = A(G-a)$ and $A'\backslash a = A(G^{ab}-a)$, therefore $r(G-a)=r(G^{ab}-a)$.

To prove the second assertion, further transform A' by adding row 1 to each row in the 3rd and 5th groups, and repeating for columns, to obtain

$$A'' = \begin{pmatrix} 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_{11} & M_{12}^c & M_{13}^c & M_{14} \\ \mathbf{0} & \mathbf{0} & M_{21}^c & M_{22} & M_{23}^c & M_{24} \\ \mathbf{0} & \mathbf{0} & M_{31}^c & M_{32}^c & M_{33} & M_{34} \\ \mathbf{0} & \mathbf{0} & M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}.$$

Because these linear transformations are all invertible, they preserve rank and nullity: $r(A)=r(A'')$. Referring to (6), note that $A''\backslash\{a,b\}$ is the adjacency matrix of $G^{ab}-a-b$. The first and second rows in A'' are linearly independent of one another and of all other rows, so deleting them reduces the rank by 2. After deletion of these rows the first two columns are all-zero, so deleting them does not change the rank; it follows that $r(A'')=r(A''\backslash\{a,b\})+2$. That is, $r(G)=r(A)=r(A'')=r(G^{ab}-a-b)+2$. ∎

Note that if either a or b has a loop, the top-left submatrix of $A(G)$ differs from that in (5), resulting in a different "border" in the matrix A'', so that the border's deletion changes the nullity unpredictably. That is, if there is a loop at either a or b, $n(G)-n(G^{ab}-a-b)$ may be 0 rather than 2. Thus, the looped case is dealt with in Section 2.2.

Theorem 3. *For any edge ab of a graph G, where a and b are both loopless,*

$$q(G) = q(G-a) + q(G^{ab}-b) + ((x-1)^2-1)q(G^{ab}-a-b).$$

Proof. For S ranging over subsets of $V(G)\backslash\{a,b\}$, by (2),

$$q(G) = \sum_S \{m(G[S]) + m(G[S\cup a]) + m(G[S\cup b]) + m(G[S\cup\{a,b\}])\} \tag{7}$$

while

$$\begin{aligned} &q(G-a) + q(G^{ab}-b) + ((x-1)^2-1)q(G^{ab}-a-b) \\ &= \sum_S \{m((G-a)[S]) + m((G-a)[S\cup b])\} \\ &\quad + \sum_S \{m(G^{ab}-b)[S]) + m((G^{ab}-b)[S\cup a])\} \\ &\quad + \sum_S ((x-1)^2-1)m((G^{ab}-a-b)[S]). \end{aligned} \tag{8}$$

To show that (7) and (8) are equal, we will show equality of their terms for each S. Two terms of (7) directly match their counterparts in (8): $m(G[S]) = m((G-a)[S])$ and $m(G[S\cup b]) = m((G-a)[S\cup b])$. A third equality follows from the first part of Lemma 2:

$$\begin{aligned} m(G[S\cup a]) &= m(G[S\cup\{a,b\}]-b) \\ &= m((G[S\cup\{a,b\}])^{ab}-b) \\ &= m(G^{ab}[S\cup a]) \qquad \text{(by (4))} \\ &= m((G^{ab}-b)[S\cup a]). \end{aligned}$$

The final equality, between a single term from (7) and two terms from (8), follows from the second part of Lemma 2:

$$\begin{aligned} m(G[S\cup\{a,b\}]) &= (x-1)^2 m((G[S\cup\{a,b\}])^{ab}-a-b) \\ &= (x-1)^2 m(G^{ab}[S]) \\ &= m(G^{ab}[S]) + ((x-1)^2-1)m(G^{ab}[S]) \\ &= m((G^{ab}-b)[S]) + ((x-1)^2-1)m((G^{ab}-a-b)[S]). \ \blacksquare \end{aligned}$$

2.2. Local complementation

In the case of graphs with loops, there may not always be an edge ab, with both a and b loopless, to which Theorem 3 (derived from Lemma 2) may be applied. In this case, though, there must be a looped vertex, and we may apply a different reduction instead.

Bouchet [9] defines the "local complement" G^a of a simple graph G on vertex a by complementing (toggling the presence or absence of all edges in) the subgraph of G induced by the neighborhood of a, while keeping the graph otherwise unchanged. We extend this to graphs with loops, toggling loops just like other edges, and with the convention that $a \notin \Gamma(a)$ even if a is looped.

Definition 4 (Local complementation). A graph G is locally complemented on a vertex a to yield G^a, where G^a is equal to G except that $G^a[\Gamma(a)] = \overline{G[\Gamma(a)]}$.

In notation like that of (5) and (6) but only distinguishing a vertex a having a loop, its neighbors, and its non-neighbors, we may write

$$A(G) = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & M_{11} & M_{12} \\ \mathbf{0} & M_{21} & M_{22} \end{pmatrix} \tag{9}$$

and

$$A(G^a) = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & M_{11}^c & M_{12} \\ \mathbf{0} & M_{21} & M_{22} \end{pmatrix}. \tag{10}$$

Incidentally, it is observed in [9] that a pivot is equal to a composition of local complementations, $G^{ab} = ((G^a)^b)^a$, followed by a swap of the labels a and b. (This is for our version of pivoting, which differs from Bouchet's by a label swap, and it holds only for simple graphs; when loops are allowed, the two quantities always differ in a loop on a or on b.)

Lemma 5. *For any graph G with a looped vertex a, $r(G) = r(G^a - a) + 1$, and equivalently $n(G) = n(G^a - a)$.*

Proof. With the adjacency matrix $A = A(G)$ represented as in (9), let A' be obtained by adding row 1 of $A(G)$ to each row in the second class, and repeating for columns; thus

$$A' = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & M_{11}^c & M_{12} \\ \mathbf{0} & M_{21} & M_{22} \end{pmatrix}.$$

Referring to (10), note that $A(G^a-a)=A(G^a)\backslash a=A'\backslash a$. The linear transformations are invertible, so $r(A')=r(A)$. The first row of A' is independent of the others, so deleting it decreases the rank by 1, and what remains of the first column is all-zero, so deleting it does not change the rank; thus $r(A'\backslash a)=r(A')-1$. We conclude that $r(G^a-a)=r(A'\backslash a)=r(A')-1=r(A)-1=r(G)-1$. ∎

Theorem 6. *For a graph G, for any edge ab where neither a nor b has a loop,*

$$q(G)=q(G-a)+q(G^{ab}-b)+((x-1)^2-1)q(G^{ab}-a-b), \tag{11}$$

and for any looped vertex a,

$$q(G)=q(G-a)+(x-1)q(G^a-a). \tag{12}$$

Proof. Equation (11) is precisely Theorem 3, repeated here to give one comprehensive theorem.

To prove (12), as in the proof of Theorem 3, we show that for each $S\subseteq V(G)\backslash a$, we have equality between the corresponding summands of

$$q(G)=\sum_S\{m(G[S])+m(G[S\cup a])\}$$

and those of

$$q(G-a)+(x-1)m(G^a-a)=\sum_S m((G-a)[S])+(x-1)\sum_S m((G^a-a)[S]).$$

The first terms, $m(G[S])$ and $m((G-a)[S])$, are identical. By Lemma 5, $m(G[S\cup a])=(x-1)\,m((G[S\cup a])^a-a)=(x-1)\,m((G^a-a)[S])$, which completes the proof. ∎

The reduction formulas give an alternative characterization of the two-variable interlace polynomial. We write E_n for the empty graph on n vertices.

Corollary 7. *The two-variable interlace polynomial defined by (1) is the unique map $q:\mathcal{G}\to\mathbb{Z}[x,y]$ that satisfies the reduction formulas (11,12) and the boundary conditions $q(E_n)=y^n$, $n=0,1,\ldots$.*

Proof. By Theorem 6, the two-variable interlace polynomial $q(G)$ defined by (1) satisfies the reduction formulas (11,12) and from (1) it is immediate that it also satisfies the boundary conditions. Uniqueness follows from (11,12) by induction on the order of the graph. ∎

3. Two properties

As will be shown in the next section, the single-variable interlace polynomial defined in [2], which there was denoted as $q(G;x)$, is a special case of the present two-variable interlace polynomial, and we will denote it here as $q_N(G;y)$. (The reason for this notation, and the definition of q_N, will be given in the next section.)

Since [2] showed that this single-variable interlace polynomial satisfies the identity $q_N(G;y)=q_N(G^{ab};y)$ (at least for loopless graphs, all that [2] considered), something of the same sort might be expected for the two-variable polynomial. In fact it is not generally true that $q(G;x,y)$ is equal to $q(G^{ab};x,y)$ – a counterexample is the path of length 3, pivoted on the middle edge – but instead we have the following proposition.

Proposition 8. *For any graph G with edge ab,*

$$q(G-a)-q(G-a-b)=q(G^{ab}-a)-q(G^{ab}-a-b).$$

Proof. By the interlace polynomial's definition, with sums taken over subsets $S\subseteq V(G)\setminus\{a,b\}$,

$$\begin{aligned} q(G-a)-q(G-a-b) &= \sum_S (m(G[S])+m(G[S\cup b]))-\sum_S m(G[S]) \\ &= \sum_S m(G[S\cup b]) \\ &= \sum_S m(G^{ab}[S\cup b]) \end{aligned}$$

by Lemma 2. From this point symmetry completes the proof. ∎

As with the earlier single-variable polynomial $q_N(G;y)$, the present two-variable polynomial obeys a simple product rule. For graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ with disjoint vertex sets, $V_1\cap V_2=\emptyset$, let $G_1\cup G_2$ denote their disjoint union, $(V_1\cup V_2,E_1\cup E_2)$.

Proposition 9. *If G_1 and G_2 are graphs on disjoint vertex sets, and $G_1\cup G_2$ their disjoint union, then $q(G_1\cup G_2)=q(G_1)q(G_2)$.*

Proof. As in [2], the relation can be proved through the reduction, taking advantage of the fact that pivots in one component do not affect the other. Alternatively, it is easy to see that the sum (1) is the product of the corresponding sums over all $S_1\subseteq V(G_1)$ and $S_2\subseteq V(G_2)$. ∎

4. Specializations of the interlace polynomial

Specializing $q(G;x,y)$ by setting $x=2$ (or, for loopless graphs, where the rank is always even, $x=0$) causes the “rank” term to disappear from (1), and so we call this polynomial the “vertex-nullity interlace polynomial”,

$$q_N(G;y) = q(G;2,y) = \sum_{S \subseteq V(G)} (y-1)^{n(G[s])}. \tag{13}$$

As alluded to in the previous section, this is the single-variable interlace polynomial studied in [2]. To see this, note that substituting $x=2$ into (11) gives the reduction $q_N(G;y) = q_N(G-a;y) + q_N(G^{ab}-b;y)$, precisely the reduction that defined the single-variable polynomial in [2] (which was defined only for loopless graphs). The boundary conditions $q_N(E_n;y) = y^n$ also match, concluding the equivalence.

Corollary 10. *The single-variable interlace polynomial of [2], defined there by a reduction formula, has the explicit expansion* (13).

This single-variable interlace polynomial $q_N(G)$ is related to the Martin polynomial and circuit partition polynomials [17,6–8,13,14,5,1]. Bouchet [10] recognized that the single-variable interlace graph polynomial of [2] was a specialization of the Tutte–Martin polynomial of an isotropic system (introduced by Bouchet in [6] and generalizing the Martin polynomial), and this connection was clarified and made explicit by Aigner and van der Holst in [3]. [3], written after the present work's submission, also proved a conjecture from [2] that $q_N(G;-1)$ is always of the form $\pm 2^s$ (previously proved in [4]), independently derived the expansion (13) of the nullity polynomial, and introduced a related polynomial. However, in contrast to the Tutte-like two-variable graph polynomial of the present paper, we are not aware of any two-variable generalization of the Tutte–Martin polynomial.

Analogously to the nullity polynomial $q_N(G;y)$, there is also a single-variable “vertex-rank interlace polynomial”

$$q_R(G;x) = q(G;x,2) = \sum_{S \subseteq V(G)} (x-1)^{r(G[S])},$$

and it appears to be equally interesting. First, the vertex-rank polynomial distinguishes graphs of small order better than the vertex-nullity polynomial. For example, the rank polynomial distinguishes all 11 simple graphs of order 4, where the nullity polynomial takes on only 8 distinct values. (And the rank polynomial distinguishes all 90 looped graphs of order 4, where the nullity polynomial takes on only 17 distinct values). At order 5 there are 34 non-isomorphic simple graphs: the rank polynomial takes 33 values, and the nullity polynomial only 17. (There are 544 non-isomorphic looped graphs, the rank polynomial takes 541 values, and the nullity polynomial only 41.) Similarly for trees: the nullity polynomial fails to distinguish one pair of trees of order 8 and two pairs of order 9; the rank polynomial distinguishes all trees of orders 8 and 9.

In [1] it was shown that certain basic graph parameters could be read out from the vertex-nullity polynomial, namely the order, the component number, the edge-independence number, and an upper bound on the (vertex) independence number. The vertex-rank polynomial, too, gives the order.

Remark 11. For any graph G of order n, $q_R(G;2)=2^n$.

Proof. The formula in (1) reduces to a sum, over all 2^n subgraphs of G, of 1 raised to a power. ∎

As per the following proposition, the maximum degree of either variable in the two-variable interlace polynomial is unchanged by "removing" the other variable (substituting 2).

Proposition 12. *For any graph G, $\deg_x(q(G;x,y)) = \deg(q_R(G;x))$, and $\deg_y(q(G;x,y))=\deg(q_N(G;y))$, where $\deg_x$ (respectively $\deg_y$) denotes the maximum degree of x (resp. y) in the polynomial.*

Proof. We will prove the statement for the vertex-rank polynomial; that for the vertex-nullity polynomial is proved identically. Since $q_R(G;x) = q(G;x,2)$, $\deg(q_R(G;x)) \leq \deg_x(q(G;x,y))$. Consider any $S \subseteq V(G)$ contributing to (1) a term of the maximum x-degree, degree k. But $q_R(G;x)=\sum_{S\subseteq V(G)}(x-1)^{r(G[S])}$, so each such term here also has x-degree k. For each such S the coefficient of x^k is 1: there is no cancellation, and so the x-degree is k in q_R as it was in q. ∎

In [1] we showed that $\deg(q_N(G)) \geq \operatorname{ind}(G)$, that is, $\deg(q_N(G))$ is an upper bound on the independence number. (While it was proved there only for loopless graphs, the same result for looped graphs follows from (14), in the next section.) We showed graphs for which the absolute gap $\deg(q_N(G))-\operatorname{ind}(G)$ was arbitrarily large, but it remained open whether the ratio $\deg(q_N(G))/\operatorname{ind}(G)$ could be made arbitrarily large. Indeed it can.

Let H_d be the d-dimensional Hamming cube (so $|H_d|=2^d$) and let $\overline{H}_d$ be its complement. Then we have the following.

Remark 13. The Hamming cube and its complement have (respectively) clique and independence numbers $\operatorname{cl}(H_d)=\operatorname{ind}(\overline{H}_d)=2$. For $d>1$ odd, their nullities are $n(H_d)=0$ and $n(\overline{H}_d)=2^d/2$, and for d even, $n(H_d)=2^d/2$ and $n(\overline{H}_d)=0$.

For $\overline{H}_d$ with $d>1$ odd, the set $S=V(\overline{H}_d)$ is a witness that $\deg(q_N(\overline{H}_d))\geq 2^d/2$, and therefore $\deg(q_N(\overline{H}_d))/\operatorname{ind}(\overline{H}_d)\geq 2^d/4$.

Proof. That $\operatorname{ind}(\overline{H}_d)=\operatorname{cl}(H_d)=2$ is immediate from the structure of the Hamming cube. We next derive the nullity of H_d, from which that of $\overline{H}_d$ follows quickly. Let the adjacency matrix of H_d be A_d and that of $\overline{H}_d$ be $\overline{A}_d$.

We first show that A_d is self-inverse, for odd d. The dot product of the ith and jth rows of A_d is the parity of the number of vertices at Hamming distance 1 to both i and j in H_d. If $i=j$, then there are d such vertices, for parity 1. If the distance between i and j is more than 2, then there are no such vertices: parity 0. And if i and j are at distance exactly 2, then the vertices in question must agree with both i and j where those two agree, agree with i in one of the two coordinates where it differs from j, and agree with j in the other; there are 2 such vertices, for parity 0. When d is even everything is the same except in the case $i=j$, where the d common neighbors mean parity 0, and thus $A_d^2=\mathbf{0}$.

The situation is similar for $\overline{A}_d$, with the parities reversed: Modulo 2, $\overline{A}_d=A_d+\mathbf{1}+I$, where $\mathbf{1}$ is the all-1 matrix. Then for d odd $\overline{A}_d^2=I$, and for d even $\overline{A}_d^2=\mathbf{0}$.

For d odd, invertibility of A_d means it is of full rank. The self-invertibility of A_d for d odd can also be used to show that A_d is of half the full rank for d even: "Gluing" together two copies of H_{d-1} to make an H_d, $A_d = \begin{pmatrix} A_{d-1} & I \\ I & A_{d-1} \end{pmatrix}$, self-invertibility of A_{d-1} means the second set of rows is simply A_{d-1} times the first set. Since the second set is a linear combination of the first set, the rank of the whole matrix is at most 2^{d-1}; the presence of a block I means this rank is achieved.

Similarly, for $\overline{H}_d$ we have $\overline{A}_d=\begin{pmatrix} \overline{A}_{d-1} & \overline{I} \\ \overline{I} & \overline{A}_{d-1} \end{pmatrix}$, where $\overline{I}=\mathbf{1}+I$. Here we find that for d odd, the second set of rows is just $\overline{A}_{d-1}\overline{I}=\overline{I}\,\overline{A}_{d-1}$ times the first set, and conclude that $r(\overline{A}_d)=2^{d-1}$. ∎

5. Counting independent sets

There are other interesting specializations of the 2-variable interlace polynomial. Evaluating at $y=1$ means that $(y-1)^{n(G[H])}=0$ except when $n(G[H])=0$ giving $(y-1)^{n(G[H])}=1$. In particular, then, $q(G;2,1)=\sum_{H\subseteq G}(2-1)^{r(H)}(1-1)^{n(H)}=\sum_{H:n(H)=0}1^{r(H)}$ counts full-rank induced subgraphs of G.

Similarly, $q(G;1,2) = \sum_{H:r(H)=0} 1^{n(H)}$ counts the independent sets of G (including the empty set), a problem that has received widespread attention. In particular, it is known that counting independent sets (computing the independence number) is #P-complete even for low-degree graphs [12], so it follows that it is #P-hard to compute the two-variable interlace polynomial (in particular at the point $(x,y) = (1,2)$) and the "rank" interlace polynomial (at $x = 1$). In fact, it is hard to compute the independence number even approximately [11], and so the interlace polynomial must be hard even to approximate at these points.

Given the similarity to the Tutte polynomial, which is hard to compute almost everywhere ([15], see also [19] for a survey), and given the variety of structures counted by the interlace polynomial (see [1]), with counting typically being #P-hard, it is anything but surprising that the interlace polynomial is computationally hard. However, it was a question left unresolved in [1], and in fact we still do not have a proof that computing the "nullity" polynomial is #P-hard. Moreover, in analogy with the Tutte polynomial, it would be of interest to show that the interlace polynomial is hard to compute at almost all points (x,y).

A particularly interesting evaluation is

$$q(G;1,1+\lambda) = \sum_{H\subseteq G} 0^{r(H)}\lambda^{n(H)} = \sum_{H:\, r(H)=0} \lambda^{|H|} = I(G;\lambda), \tag{14}$$

the independent-set polynomial (the sum over all $k \geq 0$ of λ^k times the number of independent sets of cardinality k). It is well known that further quantities of interest can be computed from the independent-set polynomial and its derivatives. For example from (14) it is clear that $\frac{\partial}{\partial\lambda} q(G;1,1+\lambda) = \sum_{H:r(H)=0} |H|\lambda^{|H|-1}$. For $\lambda = 1$ this is just $\sum_{H:r(H)=0} |H|$, so that $\frac{\partial q(G)}{\partial y}(1,2)$ is the sum of the sizes of all independent sets.

6. Polynomials of some basic graphs

We compute the interlace polynomial of some basic graphs, notably complete graphs K_n, complete bipartite graphs $K_{m,n}$, and paths P_n of length n.

Proposition 14. *For all n and m we have*

$$q(E_n) = y^n$$

$$q(K_n) = \frac{1}{2}\left(x^n + (2-x)^n\right) + \frac{1}{2}\left(\frac{y-1}{x-1}\right)\left(x^n - (2-x)^n\right)$$

$$q(K_{m,n}) = \frac{(x-1)^2}{(y-1)^2}\left((y^m-1)(y^n-1)\right) + y^m + y^n - 1$$

$$q(P_n) = \frac{1}{2}\left(y + \frac{3y+2x(x-2)}{\sqrt{1+4(y+x(x-2))}}\right)\left(\frac{1+\sqrt{1+4(y+x(x-2))}}{2}\right)^n$$
$$+ \frac{1}{2}\left(y - \frac{3y+2x(x-2)}{\sqrt{1+4(y+x(x-2))}}\right)\left(\frac{1-\sqrt{1+4(y+x(x-2))}}{2}\right)^n.$$

Proof. That $q(E_n) = y^n$ is immediate from (1) and also figured into the boundary condition in Corollary 7.

For K_n we have

$$q(K_n) = \sum_{k \le n} \binom{n}{k} (y-1)^{n(K_k)} (x-1)^{r(K_k)}$$

which, letting $\operatorname{odd}(k) = 1$ if k is odd and 0 otherwise

$$= \sum_{k \le n} \binom{n}{k} (y-1)^{\operatorname{odd}(k)} (x-1)^{k-\operatorname{odd}(k)}$$
$$= \sum_{k \text{ even}} \binom{n}{k} (x-1)^k + \frac{y-1}{x-1} \sum_{k \text{ odd}} \binom{n}{k} (x-1)^k.$$

The even and odd sums are computable from the sum and difference of $(z+1)^n = \sum \binom{n}{k} z^k$ and $(-z+1)^n = \sum \binom{n}{k} z^k (-1)^k$; substituting $z = x-1$ and simplifying gives the expression shown.

We also derive $q(K_{m,n})$ directly from (1). $K_{m,n}$ has $\binom{m}{i}\binom{n}{j}$ subgraphs $K_{i,j}$. Each such subgraph's adjacency matrix has the form

$$A(K_{i,j}) = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix},$$

whose rank is

$$r(K_{i,j}) = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ 2 & \text{if } i > 0 \text{ and } j > 0. \end{cases}$$

Then

$$q = \sum (x-1)^r (y-1)^n$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (x-1)^2 (y-1)^{i+j-2} \binom{m}{i}\binom{n}{j} + \sum_{j=1}^{n} \binom{n}{j} + \sum_{i=1}^{m} \binom{m}{i} + 1,$$

the four terms coming respectively from the cases where $i > 0$ and $j > 0$; $i = 0$ and $j > 0$; $j = 0$ and $i > 0$; and $i = j = 0$ (the null subgraph). Expanding,

$$= \frac{(x-1)^2}{(y-1)^2}\left(\sum_{i=1}^{m}(y-1)^i\binom{m}{i}\right)\left(\sum_{j=1}^{n}(y-1)^j\binom{n}{j}\right) + \sum_{j=1}^{n}(y-1)^j\binom{n}{j} + \sum_{i=1}^{m}(y-1)^i\binom{m}{i} + 1.$$

The claim for $K_{m,n}$ follows immediately.

For $G=P_n$ with $n\geq 2$, we use the reduction (11) with edge ab, where b is a leaf. Since $G-a$ is the disjoint union of P_{n-2} and E_1, $q(G-a)=yq(P_{n-2})$. Since $G^{ab}=G$, $G^{ab}-b=P_{n-1}$ and $G^{ab}-a-b=P_{n-2}$. The net result is

$$q(P_n) = (y + x^2 - 2x)q(P_{n-2}) + q(P_{n-1}).$$

Solving this recursion, with the boundary conditions $q(P_0)=q(E_1)=y$ and $q(P_1)=q(K_2)=x^2-2x+2y$, yields our formula for $q(P_n)$. ∎

7. Further polynomials

We observed in Section 1 that the interlace polynomial's expansion is similar to that of the Tutte polynomial, with two significant differences: the sum is over vertex rather than edge subsets, and the rank and nullity are those of an adjacency matrix rather than an incidence matrix.

This suggests a whole range of polynomials given by similar expansions, with the sums taken variously over vertex or edge subsets, with rank and nullity being those of an adjacency or an incidence matrix, and with the field used perhaps being other than $\mathbb{F}_2$. Of course there is no obstacle to taking a "master polynomial" summing over both vertex and edge subsets, incorporating terms for both the adjacency-matrix and the incidence-matrix rank (using four variables instead of two), and even introducing further variables to incorporate ranks computed over other fields. It would be interesting to determine which of these polynomials satisfy reductions akin to that of Theorem 3, and which ones have significance in combinatorics or other fields.

8. Open problems

As was the case with the single-variable interlace polynomial, the two-variable interlace polynomial is quite new, and there are more questions than answers. Here we simply list a few of the obvious ones.

Is $q(G)$ reconstructible, *i.e.*, given $q(G-a)$ for each vertex a, can we reconstruct $q(G)$?

What is the expectation of $q(G)$ for a random graph G?

We conjectured in [1] that the vertex-nullity polynomial's coefficient sequence might be unimodal. For the two-variable polynomial, representing the coefficients of $q(G;-x,y)$ as an array whose entry (i,j) is the coefficient of $x^i y^j$, the array's rows and columns are also unimodal, for all (loopless) graphs through order 7. (With the substitution of $-x$ for x, it is clear from Corollary 7 – but not from (1) – that the coefficients are all non-negative.) Unfortunately, there are six graphs of order 8 for which this is not the case. Since however the vertex-rank polynomial and the vertex-nullity polynomial for these six graphs *do* have unimodal coefficient sequences, it remains possible that both of these single-variable polynomials' coefficient sequences are always unimodal. (As pointed out in [1] for the nullity polynomial, though, there are reasons to be doubtful, including the 1993 counterexample to Schwärzler's similar conjecture for the Tutte polynomial [18].)

But the most promising line of inquiry is the proposal in the previous section, to generalize from the interlace polynomial (1) and the Tutte polynomial (3) to generate other interesting polynomials.

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